21 Pages

The Dynamics and Thermodynamics of
COMPRESSIBLE FLUID FLOW

VOLUME I

BEGINNING WITH A BRIEF REVIEW of the foundation concepts of fluid dynamics and thermodynamics and an introduction to the concepts of compressible flow, this volume treats one-dimensional gas dynamics, including flow in nozzles and diffusers, normal shocks, frictional flows and flows with heat transfer or energy release; the differential equations governing the two- and three-dimensional motion of a nonviscous compressible fluid; analytical methods and experimental results for subsonic, two- and three-dimensional flows; two-dimensional supersonic flows from the theoretical and practical points of view; and, in Appendices, the theory of characteristic curves and sets of numerical tables of compressible-flow functions.

VOLUME II

IN THIS VOLUME are treated three-dimensional supersonic flows past wings and bodies of revolution; hypersonic flows; flows containing both subsonic and supersonic regions; transonic flows; unsteady flows in one dimension, including continuous wave motion and moving shocks; theoretical and experimental surveys of friction and heat transfer in laminar and turbulent boundary layers for external and internal flows; and the interaction between boundary layers and shock waves.

The Dynamics and Thermodynamics of COMPRESSIBLE FLUID FLOW

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IN TWO VOLUMES

VOLUME I

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Chapter 4

ISENTROPIC FLOW

4.1. Introductory Remarks

The one-dimensional, steady-flow treatment of isentropic flow finds important applications in two kinds of problems, namely, (i) flow in ducts and (ii) flow in stream tubes.

Flow in Ducts. The flow in pipes and ducts is very often adiabatic. When the duct is short, as it is in nozzles and diffusers, the frictional effects are comparatively small, and the flow may as a first approximation be considered reversible, and, therefore, isentropic. Furthermore, since the function of nozzles and diffusers is to accelerate or decelerate a stream as efficiently as possible, the isentropic process provides a useful standard of comparison for actual nozzles and diffusers.

Flow in a Stream Tube. In virtually all problems involving flow around bodies, and in many involving flow through passages, there are elementary stream tubes which lie entirely outside the boundary layer. Viscous and heat conduction effects for such stream tubes, it appears from experiment, are negligible. Hence the equations of isentropic flow may be considered exact, unless discontinuities such as shock waves appear.

The One-Dimensional Approximation. By a one-dimensional flow we mean a flow in which all fluid properties are uniform over any cross section of the duct. Or, more strictly, we mean a flow in which the rate of change of fluid properties normal to the streamline direction is negligibly small compared with the rate of change along the streamline.

No approximation whatsoever is involved in the case of a stream tube, for the flow through an infinitesimal stream tube is, in the limit, exactly one-dimensional.

When applying the one-dimensional assumption to the flow in ducts, where it is well known that the properties vary over each cross section, we in effect deal with certain kinds of average properties for each cross section. The errors in predicting the rate of change of properties along the duct axis may be expected to be small then, if

(i) The fractional rate of change of area with respect to distance along the axis is small ($dA/A\,dx \ll 1$).

(ii) The radius of curvature of the duct axis is large compared with the passage diameter.

(iii) The shapes of the velocity and temperature profiles are approximately unchanged from section to section along the axis of the duct.

The great virtue of the one-dimensional approximation is the marvellous simplicity it affords, leading as it does to rapid calculation methods for a great variety of practical engineering problems. Indeed, the information resulting from the one-dimensional point of view is, when carefully interpreted, so useful and so reliable that this method is one of the most powerful tools in the armory of the engineer.

At the same time, it is well to remember that the one-dimensional approach, by its very nature, supplies information only as to the way in which the average fluid properties over the duct cross section vary with distance along the axis of the duct, and is completely silent as to the variation of properties normal to the streamlines. For many practical problems the latter is of the essence, and then the one-dimensional treatment must be supplemented by a two- or even three-dimensional analysis.

NOMENCLATURE

A	cross-sectional area	s	entropy per unit mass
c	speed of sound	T	absolute temperature
c_p	specific heat at constant pressure	$\mathfrak{I}$	thrust
c_v	specific heat at constant volume	V	velocity
C_p	pressure coefficient	w	mass rate of flow
C_w	nozzle discharge coefficient	W	molecular weight
F	impulse function	η	nozzle efficiency
G	mass velocity, w/A	μ	viscosity
h	enthalpy per unit mass	ρ	density
k	ratio of specific heats		
M	Mach Number	$(\ )_0$	signifies stagnation state
M^*	dimensionless speed, V/c^*	$(\ )^*$	signifies state at which the Mach Number is unity; does not apply to M^*
p	pressure		
R	gas constant		
$\mathfrak{R}$	universal gas constant	$(\ )_\infty$	signifies free stream conditions

4.2. General Features of Isentropic Flow

Let us first consider the isentropic flow of any fluid whatsoever through a passage of varying cross section (Fig 4.1). All possible states lie on a line of constant entropy, as shown in Fig. 4.2. One state on this isentropic corresponds to zero velocity. The pressure at this state, p_0, is usually called the *isentropic stagnation pressure*, and is sometimes called the *total pressure*. The *value of the stagnation enthalpy*, h_0, is independent of whether or not entropy changes occur, since it has the same value for all states which are reachable from it adiabatically.

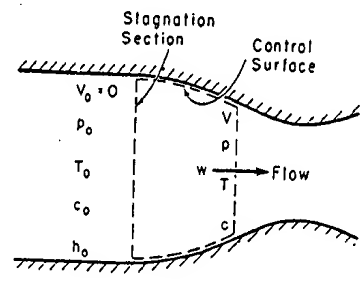


FIG. 4.1. Flow between stagnation section and any other section.

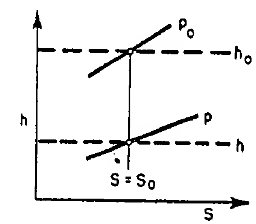


FIG. 4.2. Isentropic acceleration or deceleration on Mollier chart, showing stagnation enthalpy and isentropic stagnation pressure.

Governing Physical Equations. The following physical equations may be written for a control surface extending between the stagnation section and any other section in the channel:

First Law of Thermodynamics.

$$h_0 = h + \frac{V^2}{2} \tag{4.1a}$$

Second Law of Thermodynamics.

$$s = s_0 \tag{4.1b}$$

Equation of Continuity.

$$G = w/A = \rho V \tag{4.1c}$$

Equation of State. This may be expressed in the form of charts, tables, or algebraic equations. We may write implicitly that

$$h = h(s, p) \tag{4.1d}$$

$$\rho = \rho(s, p)$$

Definition of Mach Number.

$$M = V/c = V/\sqrt{(\partial p/\partial \rho)_s} \tag{4.1e}$$

Performance Curves. For a given stagnation condition, i.e., for fixed values of s_0 and p_0, Eqs. 4.1 may be used for constructing the performance curves shown in Fig. 4.3. For example, suppose an arbitrary value of p less than p_0 is selected. Then the corresponding values of h and ρ

may be found from Eq. 4.1d, inasmuch as s is known from Eq. 4.1b; the corresponding value of V may next be found from Eq. 4.1a; and, finally, the corresponding values of w/A and M may be reckoned from Eqs. 4.1c and 4.1e.

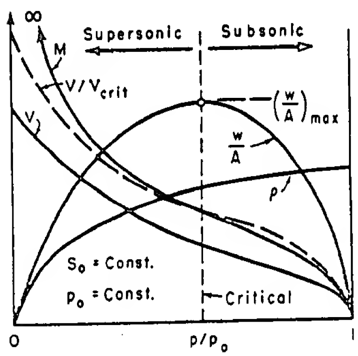


FIG. 4.3. Typical variation of flow properties in isentropic flow.

The curves shown in Fig. 4.3 are typical of gases and vapors but are somewhat different for liquids and liquid–vapor mixtures.

Critical Pressure Ratio. The most interesting feature of Fig. 4.3 is the maximum in the curve of flow per unit area, which indicates that an accelerating stream starting from rest must first decrease in cross section and then subsequently increase in cross section. The pressure ratio, p/p_0, where the flow per unit area is a maximum, is called the *critical pressure ratio*, and has a value, for all real gases and vapors, of approximately one-half.

Pressure ratios greater than the critical correspond to subsonic flow, and pressure ratios less than the critical correspond to supersonic flow. This will now be demonstrated in a completely general manner.

Distinction Between Subsonic and Supersonic Flow. We first write the steady-flow energy equation in differential form for two cross sections infinitesimally distant from each other. Thus

$$dh = -d(V^2/2) = -V\,dV \tag{4.2a}$$

From the thermodynamic relation, $T\,ds = dh - dp/\rho$, and the condition of constant entropy, we have

$$dh = dp/\rho \tag{4.2b}$$

so that

$$dp = -\rho V\,dV \tag{4.2c}$$

This will be recognized as Euler's equation of motion for an inviscid fluid. This is not surprising, as the kinetic-energy term in the steady-flow energy equation was originally obtained with the help of Newton's second law of motion.

Next we introduce the equation of continuity in logarithmic differential form,

$$d(\ln \rho A V) = 0; \quad \text{or} \quad d(\ln \rho) + d(\ln A) + d(\ln V) = 0$$

giving

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \tag{4.2d}$$

Substituting Eq. 4.2c into Eq. 4.2d and rearranging, we get

$$\frac{dA}{A} = \frac{dp}{\rho}\left(\frac{1}{V^2} - \frac{d\rho}{dp}\right)$$

Since the process is isentropic,

$$dp/d\rho = (\partial p/\partial \rho)_s = c^2$$

so that

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left(1 - \frac{V^2}{c^2}\right) = \frac{1 - M^2}{\rho V^2}\,dp \tag{4.2e}$$

Now, from Eq. 4.2c, which is the dynamic equation for a frictionless fluid, it is seen that the pressure always decreases in an accelerating flow and increases in a decelerating flow. In other words,

$$\frac{dV}{dp} < 0$$

Using this result in conjunction with Eq. 4.2e, we arrive at the following conclusions of practical significance:

(i) *For subsonic speeds* (M < 1),

$$\frac{dA}{dp} > 0; \quad \frac{dA}{dV} < 0$$

(ii) *For supersonic speeds* (M > 1),

$$\frac{dA}{dp} < 0; \quad \frac{dA}{dV} > 0$$

(iii) *For sonic speeds* (M = 1),

$$\frac{dA}{dp} = 0; \quad \frac{dA}{dV} = 0$$

Thus, we have the astonishing result that the effects of an area change, say an increase in area, are exactly opposite for subsonic and supersonic flow.

The possible types of flow, according to this tabulation, are summarized schematically in Fig. 4.4.

At Mach Number unity the area goes through a minimum. This important conclusion is valid irrespective of the type of fluid considered, whether gaseous or liquid.

In constructing the curves of Fig. 4.3, the equation of state, Eq. 4.1d, was employed. Since the equation of state for real gases and vapors can seldom be put into simple algebraic form, the curves of Fig. 4.3 cannot, in general, be formulated analytically, but instead are found through

direct computation. If, on the other hand, it is assumed that the perfect gas laws are valid, analytical results are obtainable, and the numerical computation of problems is greatly simplified. For many engineering gases, particularly air at moderate pressures and temperatures, the deviations from the perfect gas laws are negligible; hence, most calculations are based on these simple relations.

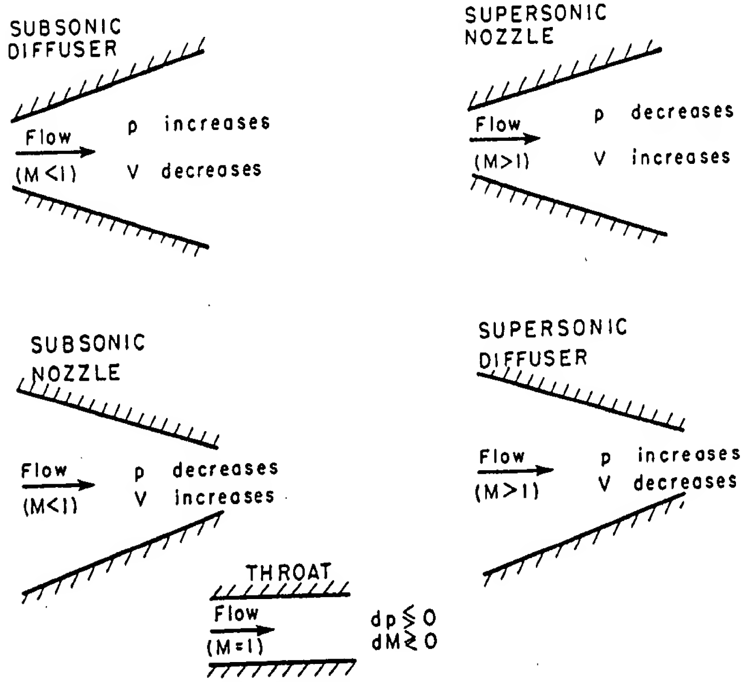


FIG. 4.4. Effects of area change on pressure and velocity in subsonic and supersonic flow.

4.3. Adiabatic Flow of a Perfect Gas

Before restricting the discussion to isentropic flow, certain relations obtainable from the energy equation alone will be derived. These relations are valid for any adiabatic flow of a perfect gas, whether reversible or not.

For a perfect gas we have

$$\Delta h = c_p \, \Delta T \tag{4.3a}$$

$$c_p - c_v = R \tag{4.3b}$$

$$c_p/c_v = k \tag{4.3c}$$

$$c_p = \frac{k}{k-1} R \tag{4.3d}$$

Using Eq. 4.3a and 4.3d, the steady-flow energy equation (Eq. 4.1a) becomes

$$V = \sqrt{2c_p(T_0 - T)} = \sqrt{\frac{2k}{k-1} R(T_0 - T)} \tag{4.4}$$

From this we see that for a fixed *stagnation temperature* T_0, (sometimes called *total temperature*), all states with the same temperature have the same velocity. Referring to the temperature-entropy diagram of Fig. 4.5, lines of constant velocity are horizontal, and the vertical distance between T_0 and T is proportional to the square of the velocity.

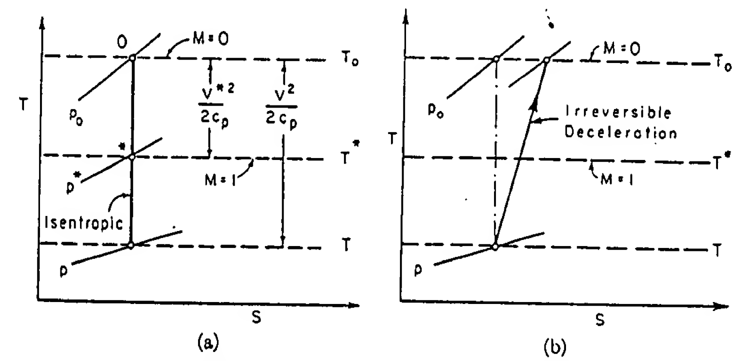


FIG. 4.5. Flow processes on temperature-entropy diagram.
(a) Isentropic acceleration or deceleration.
(b) Irreversible adiabatic deceleration.

Three Reference Speeds. Since negative temperatures on the absolute scale are not attainable, it is evident from Eq. 4.4 that there is a maximum velocity corresponding to a given stagnation temperature. This maximum velocity, which is often used for reference purposes, is given by

$$V_{\max} = \sqrt{\frac{2k}{k-1} RT_0} \tag{4.5a}$$

Another useful reference velocity is the speed of sound at the stagnation temperature,

$$c_0 = \sqrt{kRT_0} \tag{4.5b}$$

Still a third convenient reference velocity is the critical speed, i.e., the velocity at Mach Number unity. Using an asterisk to denote conditions at $M = 1$, we have, by definition,

$$V^* \equiv c^*$$

or, using Eq. 4.4 and the equation for the sound speed in a perfect gas,

$$\sqrt{\frac{2k}{k-1}R(T_0 - T^*)} = \sqrt{kRT^*}$$

which gives, after rearrangement

$$\frac{T^*}{T_0} = \frac{2}{k+1}$$

Substituting this value of T^* for T in Eq. 4.4, we get

$$V^* = c^* = \sqrt{\frac{2k}{k+1}RT_0} \tag{4.5c}$$

Now, using Eqs. 4.5, we get the following relations between the three reference velocities, together with the numerical values for $k = 1.4$:

$$\frac{c^*}{c_0} = \sqrt{\frac{2}{k+1}} = 0.913 \tag{4.6a}$$

$$\frac{V_{max}}{c_0} = \sqrt{\frac{2}{k-1}} = 2.24 \tag{4.6b}$$

$$\frac{V_{max}}{c^*} = \sqrt{\frac{k+1}{k-1}} = 2.45 \tag{4.6c}$$

Stagnation-Temperature Ratio. Eq. 4.4 may be written

$$\frac{T_0}{T} = 1 + \frac{V^2}{2c_pT} = 1 + \frac{V^2}{kRT}\frac{kR}{2c_p}$$

Since $c_p = kR/(k-1)$ and $c^2 = kRT$, this takes the simple and convenient form

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\frac{V^2}{c^2} = 1 + \frac{k-1}{2}M^2 \tag{4.7}$$

which shows that the *stagnation-temperature ratio* depends only on the gas constant k and Mach Number M.

The Energy Equation in Kinematic Form. The energy equation for adiabatic flow of a perfect gas is

$$V^2 + 2c_pT = \text{constant}$$

Moreover, for a perfect gas,

$$c_pT \equiv \frac{c_p}{kR}kRT = \frac{1}{k-1}c^2$$

Combining these relations, and evaluating the constant at the three reference conditions of (i) zero speed, (ii) zero temperature, and (iii) sonic speed, we obtain three alternate and useful forms of the energy equation involving only the local velocity, local sound speed, and k:

$$V^2 + \frac{2}{k-1}c^2 = \frac{2}{k-1}c_0^2 = V_{max}^2 = \frac{k+1}{k-1}c^{*2} \tag{4.8}$$

Note that Eqs. 4.5 through 4.7 may be obtained rather easily from Eq. 4.8.

The Dimensionless Velocity M^*. As a dimensionless parameter the Mach Number is very convenient, but it has two disadvantages: (i) it is not proportional to the velocity alone and, (ii) at high speeds it tends towards infinity. Often, therefore, it is useful to work with a dimensionless velocity obtained through dividing the flow velocity V by one of the three reference velocities of Eq. 4.8. Generally the most useful of these ratios is defined by

$$M^* \equiv V/c^* \equiv V/V^*$$

It should be noted immediately that although in general the asterisk denotes the value of a property at Mach Number unity, this convention is not followed in the definition of M^*. The latter is *not* the value of M at the local sonic condition, but is rather defined as given above.

There is a unique relation between M and M^* for adiabatic flow. From the definitions of M^* and M,

$$M^{*2} \equiv \frac{V^2}{c^{*2}} = \frac{V^2}{c^2}\frac{c^2}{c^{*2}} = M^2\frac{c^2}{c^{*2}}$$

Furthermore, the first and last parts of Eq. 4.8 may be divided by c^{*2} to give

$$\frac{V^2}{c^{*2}} + \frac{2}{k-1}\frac{c^2}{c^{*2}} = \frac{k+1}{k-1}$$

Eliminating c^2/c^{*2} from this pair of equations, and rearranging, we get the useful formulas

$$M^{*2} = \frac{\frac{k+1}{2}M^2}{1 + \frac{k-1}{2}M^2} \tag{4.9}$$

and

$$M^2 = \frac{\frac{2}{k+1}M^{*2}}{1 - \frac{k-1}{k+1}M^{*2}} \tag{4.10}$$

The value of M^* is a simple index of when the flow is subsonic and when supersonic, for we find from Eqs. 4.9 and 4.10 that

$$\text{when } M < 1, \text{ then } M^* < 1$$
$$\text{when } M > 1, \text{ then } M^* > 1$$
$$\text{when } M = 1, \text{ then } M^* = 1$$
$$\text{when } M = 0, \text{ then } M^* = 0$$
$$\text{when } M = \infty, \text{ then } M^* = \sqrt{\frac{k+1}{k-1}}$$

Fig. 4.3 illustrates the relative magnitudes of M and M^* in subsonic and supersonic flow.

Flow per Unit Area. Next we will derive a useful relation between the flow per unit area, stagnation temperature, static pressure and Mach Number. Starting with the equation of continuity, we make the following rearrangements:

$$\frac{w}{A} = \rho V = \frac{p}{RT} V = \frac{pV}{\sqrt{kRT}} \sqrt{\frac{k}{R}} \sqrt{\frac{T_0}{T}} \frac{1}{\sqrt{T_0}}$$
$$= \sqrt{\frac{k}{R}} \frac{p}{\sqrt{T_0}} M \sqrt{1 + \frac{k-1}{2} M^2}$$

This is best arranged in the form of a mass flow parameter involving T_0, p, and the molecular weight W. The parameter itself then depends only on k and M according to the relation

$$\frac{w}{A} \frac{\sqrt{T_0}}{p} \frac{1}{\sqrt{W}} = \sqrt{\frac{k}{\mathcal{R}}} M \sqrt{1 + \frac{k-1}{2} M^2} \quad (4.11)$$

The parameter is plotted in Fig. 4.6 for $k = 1.4$. With this chart it is easy to solve the common problem of computing the local Mach Number from given values of w, A, T_0, and p.

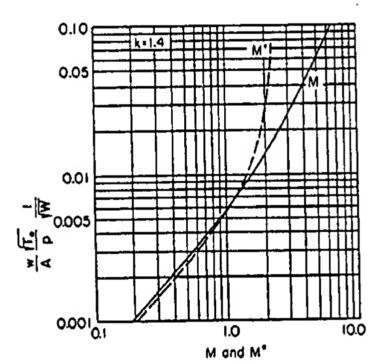


Fig. 4.6. Mass flow parameter versus M and M^*. *Units:* w (slug/sec); A (ft^2); T_0 (deg. Rankine); p (lbf/ft^2); W (lb/lb mol, or slug/slug mol).

4.4. Isentropic Flow of a Perfect Gas

All the relations of the preceding section are valid for both isentropic and nonisentropic flows. Referring to Fig. 4.5, they are applicable to any sort of change in the vertical (T) coordinate, irrespective of changes in the horizontal (s) coordinate, provided that all states have the same stagnation temperature.

We now further restrict the analysis to the isentropic case. All states along the channel or stream tube lie on a line of constant entropy and have the same stagnation temperature. The state of zero velocity is called the *isentropic stagnation state*, and the state with $M = 1$ is called the *critical state*. All states with the same entropy and the same stagnation temperature have the same isentropic stagnation state and the same critical state. When a stream with a given pressure, temperature, and velocity (Fig. 4.5b) is decelerated to zero velocity, the final pressure will be less than the isentropic stagnation pressure if the decleration is irreversible, but the final temperature will be equal to the adiabatic stagnation temperature for either reversible or irreversible deceleration.

The relations between pressure, temperature, and density for an isentropic process of a perfect gas are

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^k; \quad \frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}} \quad (4.12)$$

Also, the pressure-temperature density relation of a perfect gas is

$$\frac{p}{\rho T} = \frac{p_0}{\rho_0 T_0} = R \quad (4.13)$$

Temperature, Pressure, and Density Ratios as Functions of Mach Number. Substitution of Eqs. 4.12 into Eq. 4.7 now yields the important relations

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \quad (4.14a)$$

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \quad (4.14b)$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}} \quad (4.14c)$$

The particular values of the temperature, pressure, and density ratios at the critical state (i.e., at the minimum area) are found by setting

$M = 1$ in the above expressions. The resulting formulas, together with the numerical values for $k = 1.4$, are as follows:

$$\frac{T^*}{T_0} = \frac{c^{*2}}{c_0^2} = \frac{2}{k+1} = 0.8333 \tag{4.15a}$$

$$\frac{p^*}{p_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} = 0.5283 \tag{4.15b}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} = 0.6339 \tag{4.15c}$$

Thus, the temperature at the throat is only about 17 per cent less than the stagnation temperature, whereas the throat pressure is only about half the isentropic stagnation pressure.

The critical pressure ratio, p^*/p_0, is of the same order of magnitude for all gases. It varies almost linearly with k from 0.6065 for $k = 1$ to 0.4867 for $k = 1.67$.

Mass Flow Relations in Terms of Mach Number. From Eqs. 4.14 and 4.10 it is clear that either T/T_0, p/p_0, ρ/ρ_0, M or M^* may be taken as an independent parameter, and that the remaining quantities would then depend uniquely on the value of the chosen independent parameter. By and large, the variable M has been found to be the most convenient choice as far as simplicity of practical calculations is concerned. We shall, therefore, follow the practice in this and succeeding chapters of deriving all the working formulas in terms of M as the independent variable.

To find a convenient formula for the mass flow per unit area in terms of M, we eliminate p in the equation preceding Eq. 4.11 by means of the isentropic law (Eq. 4.14b). Thus we obtain

$$\frac{w}{A} = \sqrt{\frac{k}{R}}\frac{p_0}{\sqrt{T_0}}\frac{M}{\left(1 + \frac{k-1}{2}M^2\right)^{\frac{k+1}{2(k-1)}}} \tag{4.16}$$

This shows that, for a given Mach Number, the flow is proportional to the stagnation pressure and inversely proportional to the square root of the stagnation temperature. For this reason, flow test data on compressors and turbines, or indeed on any flow passage which operates over a wide range of pressure and temperature levels, are usually plotted with $w\sqrt{T_0}/p_0$ as the flow variable. In this way the results of a given test become applicable to operation at levels of temperature and pressure different from the original test conditions.

Now it is evident that if M were eliminated from Eq. 4.16 with the help of Eq. 4.14b, we would have w/A in terms of k, R, p_0, T_0, and p/p_0. The resulting expression is then the algebraic formula, for a perfect gas, corresponding to the curve of w/A versus p/p_0 in Fig. 4.3.

Maximum Flow per Unit Area. To find the condition of maximum flow per unit area, we could compute the derivative $d(w/A)/d(p/p_0)$ and set this derivative equal to zero. From this we would find that the critical pressure ratio is given by Eq. 4.15b.

An equivalent procedure would be to differentiate Eq. 4.16 with respect to M and set this derivative equal to zero. At this condition, we would find that $M = 1$.

However, neither of these procedures is necessary inasmuch as we have proved quite generally in Art. 4.2 that the cross-sectional area for isentropic flow passes through a minimum at Mach Number unity. Therefore, to find $(w/A)_{max}$, we need only set $M = 1$ in Eq. 4.16. Thus we find

$$\left(\frac{w}{A}\right)_{max} = \frac{w}{A^*} = \sqrt{\frac{k}{R}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}\frac{p_0}{\sqrt{T_0}} \tag{4.17}$$

For a given gas, therefore, the maximum flow per unit area depends only on the ratio $p_0/\sqrt{T_0}$. For given values of the stagnation pressure and stagnation temperature and for a passage with a given minimum area, Eq. 4.17 shows that the maximum flow which can be passed is relatively large for gases of high molecular weight and relatively small for gases of low molecular weight. Doubling the pressure level doubles the maximum flow, whereas doubling the absolute temperature level reduces the maximum flow by about 29 per cent.

Fliegner's Formula. Using the values $k = 1.4$ and $R = 53.3$ ft lbf/lbm°R, corresponding to air, we obtain from Eq. 4.17

$$\frac{w}{A^*}\frac{\sqrt{T_0}}{p_0} = 0.532 \tag{4.18}$$

for air, where w is in lbm/sec, A^* in ft^2, T_0 in °R, and p_0 in lbf/ft^2.

This formula, which we have derived on purely analytical grounds, was discovered empirically by Fliegner nearly a century ago at a time when the theoretical considerations outlined here were scarcely understood! Fliegner's experiments, which were conducted on a simple converging nozzle, gave a value of the constant within about 1 per cent of the value in Eq. 4.18.

The Area Ratio. Just as we have found it convenient to work with the dimensionless ratios p/p_0, etc., so it is convenient to introduce a

dimensionless area ratio. Obviously the appropriate reference area is A^*, and so we compute from Eqs. 4.16 and 4.17 the formula

$$\frac{A}{A^*} = \frac{w/A^*}{w/A} = \frac{1}{M}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}M^2\right)\right]^{\frac{k+1}{2(k-1)}} \qquad (4.19)$$

The area ratio is always greater than unity, and for any given value of A/A^* there always correspond two values of M—one for subsonic flow, and the other for supersonic flow.

The Impulse Function. For problems involving jet propulsion it is sometimes convenient to employ a quantity called the impulse function, defined by

$$F \equiv pA + \rho AV^2 \qquad (4.20)$$

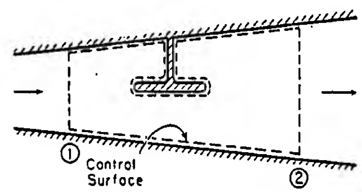


Fig. 4.7. Illustrating use of impulse function.

Applying the momentum equation to the flow through the control surface of Fig. 4.7, it is seen that

$$\mathfrak{I} + p_1A_1 - p_2A_2 = \rho_2A_2V_2^2 - \rho_1A_1V_1^2$$

or

$$\mathfrak{I} = (p_2A_2 + \rho_2A_2V_2^2) - (p_1A_1 + \rho_1A_1V_1^2) = F_2 - F_1 \qquad (4.21)$$

where $\mathfrak{I}$ is the net "thrust" produced by the stream between sections 1 and 2. The term thrust, as used here, is defined as the net force exerted by the stream on the internal solid surfaces which the fluid wets, acting in the direction opposite to the direction of flow. It includes the forces due to pressure and viscous stresses on the duct walls as well as the total drag of any stationary obstacles in the stream. This is true whether the flow is adiabatic or nonadiabatic, and whether the flow is reversible or irreversible.

For a perfect gas,

$$\rho V^2 = \frac{p}{RT}V^2 = \frac{p}{kRT}kV^2 = kpM^2$$

and so

$$F = pA(1 + kM^2) \qquad (4.22)$$

For isentropic flow, a dimensionless impulse function is formed by writing

$$\frac{F}{p_0A^*} = \frac{p}{p_0}\frac{A}{A^*}(1 + kM^2) \qquad (4.23)$$

where p/p_0 and A/A^* are functions of M given respectively by Eqs. 4.14b and 4.19.

Another way of forming a dimensionless impulse function is by evaluating F^* at M = 1, and setting

$$\frac{F}{F^*} = \frac{p}{p^*}\cdot\frac{A}{A^*}\cdot\frac{1+kM^2}{1+k} = \frac{p}{p_0}\cdot\frac{p_0}{p^*}\cdot\frac{A}{A^*}\cdot\frac{1+kM^2}{1+k}$$

substituting p/p_0, p_0/p^*, and A/A^* from Eqs. 4.14b, 4.15b, and 4.19, respectively, there is obtained after simplification,

$$\frac{F}{F^*} = \frac{1+kM^2}{M\sqrt{2(k+1)\left(1+\frac{k-1}{2}M^2\right)}} \qquad (4.24)$$

4.5. Working Charts and Tables for Isentropic Flow

Since the formulas thus far derived lead to tedious numerical calculations, often of a trial-and-error nature, practical computations are greatly facilitated by working charts and tables

Chart for Isentropic Flow. Fig. 4.8 represents in graphical form the various dimensionless ratios for isentropic flow with M as independent variable. Since changes of fluid properties in isentropic flow are brought about through changes in cross-sectional area, the key curve on this chart is that of A/A^*. The effects of changes in area on other properties may easily be found by tracing the curve of A/A^*, keeping in mind that A^*, p_0, F^*, etc. are all constant reference values for a given problem. For example, an increase in area at subsonic speed produces a decrease in velocity, an increase in F, and increases in p, T, and ρ.

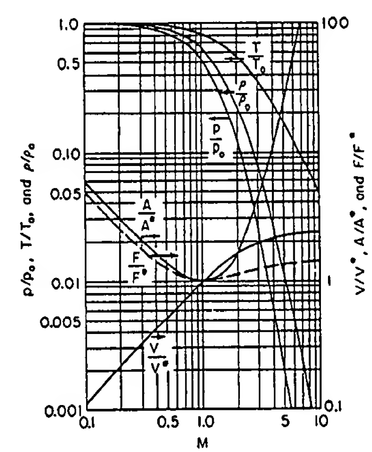


Fig. 4.8. Working chart for isentropic flow, with $k = 1.4$.

It may be seen from Fig. 4.8 that up to a Mach Number of about 0.3 (corresponding to about 300 ft/sec for air at normal conditions), changes in density are almost negligible for engineering calculations. This explains why in so many instances air is treated as though it were an incompressible fluid.

Working Tables. For accurate or extensive calculations, Table B.2 lists the various isentropic flow functions for $k = 1.4$, with Mach Number as independent argument.

Illustrative Example. The use of the compressible flow functions is best explained by an example.

PROBLEM. A supersonic wind-tunnel nozzle is to be designed for $M = 2$, with a throat section 1 sq ft in area. The supply pressure and temperature at the nozzle inlet, where the velocity is negligible, are 10 psia and 100°F, respectively. The preliminary design is to be based on the assumptions that the flow is isentropic, with $k = 1.4$, and that the flow is one-dimensional at the throat and test section. It is desired to compute the mass flow, the test-section area, and the fluid properties at the throat and test section.

SOLUTION. Table B.2 is to be used. First we work out the reference stagnation properties. We are given $p_0 = 10$ psia and $T_0 = 459.7 + 100 = 559.7$°R. From these we compute

$$\rho_0 = \frac{p_0}{RT_0} = \frac{10(144)}{53.3(559.7)} = 0.0483 \text{ lbm/ft}^3$$

$$c_0 = 49.1\sqrt{T_0} = 49.1\sqrt{559.7} = 1162 \text{ ft/sec}$$

Next, we find the properties at the throat by selecting values from Table B.2 at $M = 1$:

$$p^*/p_0 = 0.528; \qquad \therefore\ p^* = 10(0.528) = 5.28 \text{ psia}$$

$$T^*/T_0 = 0.833; \qquad \therefore\ T^* = 0.833(559.7) = 466\text{°R}$$

$$\rho^*/\rho_0 = 0.634; \qquad \therefore\ \rho^* = 0.634(0.0483) = 0.0306 \text{ lbm/ft}^3$$

$$c^*/c_0 = \sqrt{T^*/T_0} = 0.913; \qquad \therefore\ c^* = V^* = 0.913(1162) = 1060 \text{ ft/sec}$$

Entering Table B.2 at $M = 2$, we now calculate properties in the test section:

$$M^* = V/c^* = 1.6330; \qquad \therefore\ V = 1.633(1060) = 1731 \text{ ft/sec}$$

$$p/p_0 = 0.1278; \qquad \therefore\ p = 0.1278(10) = 1.278 \text{ psia}$$

$$\rho/\rho_0 = 0.2300; \qquad \therefore\ \rho = 0.2300(0.0483) = 0.01110 \text{ lbm/ft}^3$$

$$T/T_0 = 0.556; \qquad \therefore\ T = 0.556(559.7) = 311\text{°R}$$

$$A/A^* = 1.6875; \qquad \therefore\ A = 1.6875(1) = 1.6875 \text{ ft}^2$$

Finally, we compute the mass flow from Eq. 4.18:

$$w = \frac{0.532 p_0 A^*}{\sqrt{T_0}} = \frac{0.532(10)(144)(1)}{\sqrt{559.7}} = 32.4 \text{ lbm/sec}$$

Alternatively, the mass flow may be computed from the continuity equation at the throat or test section. For example,

$$w = \rho^* A^* V^* = 0.0306(1)(1060) = 32.4 \text{ lbm/sec}$$

4.6. Choking in Isentropic Flow

The fact that the curve of mass flow per unit area has a maximum is connected with an interesting and important effect called choking.

Let us consider two sections of a stream tube having a ratio of areas A_2/A_1, and let us specify all flow properties at section 1, such as p_1, T_1, M_1, etc. From the tables or graphs we can then solve for the properties at section 2, except as discussed later. For example, corresponding to M_1 we may find in the tables $(p/p_0)_1$, $(T/T_0)_1$, and $(A/A^*)_1$. Then, since A^* is constant during the process, we may write

$$\frac{A_2}{A_1} = \frac{(A/A^*)_2}{(A/A^*)_1}$$

and so we may compute $(A/A^*)_2$. Returning to the tables, we then obtain at this value of $(A/A^*)_2$ the corresponding values of M_2, $(p/p_0)_2$, $(T/T_0)_2$, etc. Since p_0 and T_0 are constant, this allows us to compute p_2 and T_2 from

$$\frac{p_2}{p_1} = \frac{(p/p_0)_2}{(p/p_0)_1}; \quad \frac{T_2}{T_1} = \frac{(T/T_0)_2}{(T/T_0)_1}$$

Now, let us consider a passage with a given area ratio A_2/A_1 and compute in the manner outlined above the values of M_2 corresponding to several values of M_1. The results may then be plotted as in Fig. 4.9. Examination of this chart indicates two peculiarities:

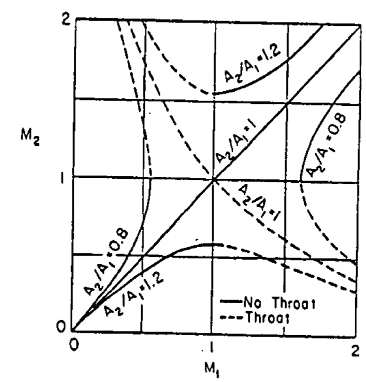


FIG. 4.9. Typical curves of M_2 versus M_1 for fixed values of area ratio A_2/A_1.

(i) For a given initial Mach Number M_1 and a given area ratio A_2/A_1, there are either two solutions for the final state M_2 or none at all. When there are two solutions, one of the two is subsonic and the other is supersonic. Which one of the two occurs depends in part on whether a throat intervenes between sections 1 and 2, for we have demonstrated that in order to go from supersonic speed to subsonic speed, or vice versa, it is necessary for the flow to pass through a throat at $M = 1$. For example, if M_1 is subsonic and the passage is converging, then M_2 must also be subsonic. On the other hand, if M_1 is subsonic and the passage is converging-diverging (i.e., has a minimum area between sections 1 and 2) the flow at section 2

may be either subsonic, as in a conventional venturi, or supersonic, as in a supersonic nozzle; which of these two situations prevails depends on the pressures imposed at the inlet and exit of the passage, as discussed more fully in Art. 4.7.

(ii) When, for selected values of M_1 and A_2/A_1, there is no solution in Fig. 4.9, the solution is imaginary in the mathematical sense. This occurs only when A_2 is smaller than A_1. Physically, this result signifies that for given conditions at section 1, there is a maximum contraction which is possible; this maximum contraction corresponds to sonic velocity at section 2. Or, put quite simply, if conditions at section 1 are specified, the mass flow is accordingly determined, and there is then a minimum cross-sectional area required to pass this flow. This phenomenon is called choking, and may be summarized by saying that for a given area reduction, there is in subsonic flow a maximum initial Mach Number which can be maintained steadily; and in supersonic flow a minimum initial Mach Number which can be maintained steadily. At either of these limiting conditions, the flow at section 2 is sonic, and is said to be choked.

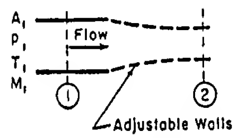


Fig. 4.10. Illustrates choking of flow.

To illustrate further the phenomenon of choking, let us suppose that at a section 1 in a duct there is a subsonic flow with certain values of M_1, p_1, T_1, and A_1. These parameters fix the flow rate, w. Let us imagine further that, at a section 2 downstream, the walls are flexible; thus the area A_2 is adjustable, as shown in Fig. 4.10. If A_2 is equal to A_1, all conditions at section 2 will be identical with the corresponding conditions at section 1. A slight reduction in A_2 will produce certain effects at section 2, which, according to Fig. 4.8, will comprise an increase in M_2, a decrease in p_2, and a decrease in T_2. This slight reduction in A_2 without a change in conditions at section 1 must, therefore, be accompanied by a reduction in the back pressure, p_2, according to the requirements of Fig. 4.8. Further reductions in A_2 may be made in the same way until the value of M_2 reaches unity.

After this point has been reached, there is no way of reducing the area further without a simultaneous change in the steady-state conditions at section 1. If, for example, the pressure and temperature at section 1 are maintained constant, a reduction in A_2/A_1 beyond its limiting value will, after a transient period of wave propagation, result in a reduced steady-state M_1, which in turn means that the flow rate will be decreased. The maximum possible value of M_1 (which will correspond to the maximum possible flow rate) is obtained when $M_2 = 1$. To obtain this limiting flow, the back pressure, p_2, must of course be adjusted accordingly. Fig. 4.8 shows that any area reduction whatsoever may be made if the initial Mach Number is sufficiently low or sufficiently high.

4.7. Operation of Nozzles Under Varying Pressure Ratios

The phenomenon of choking discussed above may be manifested in several different ways. To illustrate still another aspect of choking, let us discuss the practical problem of nozzles operating under varying pressure ratios.

Converging Nozzles. Suppose, for the sake of concreteness, that a converging passage (Fig. 4.11a) with a large entrance area at section 0

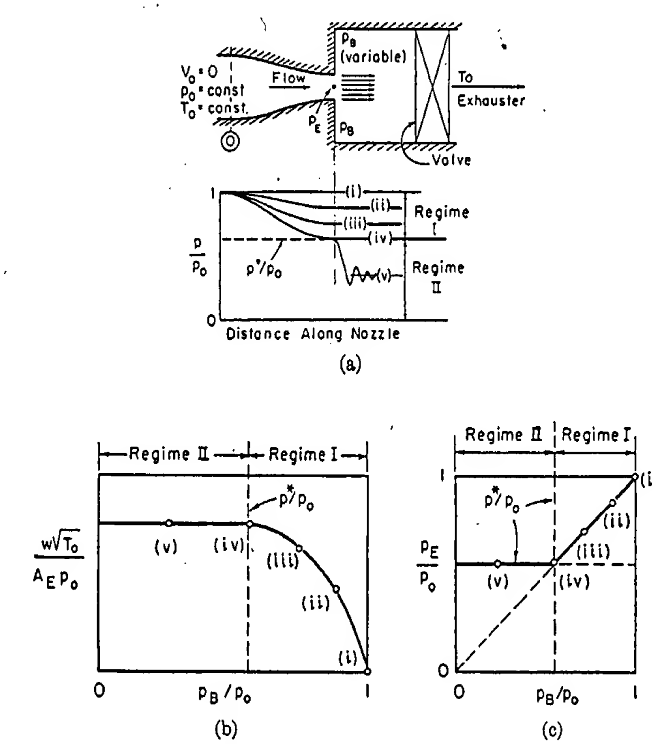


Fig. 4.11. Operation of converging nozzle at various back pressures.

discharges into a region where the back pressure, p_B, is controllable by means of a valve. The values of p_0 and T_0 will be maintained constant, and the experiment will involve variations in p_B. If p_E denotes the pressure in the exit plane of the nozzle, we inquire as to the effects of variations in back pressure on the distribution of pressure in the passage, on the flow rate, and on the exit-plane pressure. These effects are portrayed graphically in Figs. 4.11a, b, and c, respectively.

To begin with, suppose that $p_B/p_0 = 1$, shown as condition (i) in Fig. 4.11a. The pressure is then constant through the nozzle, and there is no flow.

If p_B is now reduced to a value slightly less than p_0, as shown by condition (ii), there will be flow with a constantly decreasing pressure through the nozzle. Because the exit flow is subsonic, the exit-plane pressure p_E must be the same as the back pressure p_B, except for minor secondary circulation effects in the exhaust space. That this must be so can be seen by supposing for a moment that p_E is substantially larger than p_B. If this were so the stream would expand laterally upon leaving the nozzle; however, such an area increase at subsonic speeds causes the stream pressure to rise even further. Since the back pressure is, by definition, the pressure which the stream ultimately achieves in the exhaust space, it follows that p_E cannot be larger than p_B. A similar argument leads to the conclusion that p_E cannot be substantially less than p_B.

A further reduction in p_B to condition (iii) acts to increase the flow rate and to change the pressure distribution, but there is no qualitative change in performance.

Similar considerations apply until condition (iv) is reached, at which point p_B/p_0 equals the critical pressure ratio and the value of M_E equals unity.

Further reductions in p_B/p_0, say to condition (v), cannot produce further changes in conditions within the nozzle, for the value of p_E/p_0 cannot be made less than the critical pressure ratio unless there is a throat upstream of the exit section (it is assumed here that the stream fills the passage). Consequently, at condition (v), the pressure distribution within the nozzle, the value of p_E/p_0, and the flow rate, are all identical with the corresponding quantities for condition (iv). The pressure distribution outside the nozzle cannot be predicted on one-dimensional grounds, and for the present is shown in Fig. 4.11a as a wavy curve.

To summarize the preceding discussion, the two different types of flow will be denoted as regime I and regime II. These regimes may be compared as follows:

Regime I	Regime II
$p_B/p_0 > p^*/p_0$	$p_B/p_0 < p^*/p_0$
$p_E/p_0 \cong p_B/p_0$	$p_E/p_0 = p^*/p_0$
$M_E < 1$	$M_E = 1$
$\dfrac{w\sqrt{T_0}}{A_E p_0}$ dependent on p_B/p_0	$\dfrac{w\sqrt{T_0}}{A_E p_0}$ independent of p_B/p_0

In regime I the values of p_E and p_B are virtually identical. Hence, except for a constant multiplier, the flow curve in regime I of Fig. 4.11b is identical with the curve of w/A in the subsonic part of Fig. 4.3.

A simple converging nozzle of the type discussed often serves as a flow nozzle. It is particularly useful when p_B/p_0 is less than the critical pressure ratio, for then the flow rate is given by Eq. 4.18, and measurements of only p_0, T_0, and A_E are necessary for computation of the flow rate.

For accurate measurements, the effects of boundary layer and of departures from one-dimensionality require that the nozzle be calibrated. Discharge coefficients for rounded-entrance nozzles are usually of the order 0.98 to 0.99, except for very low Reynolds Numbers where they may be considerably less.

The converging nozzle may occasionally be used to advantage as a simple flow regulator because of the fact that the flow rate is independent of back pressure when the latter is less than about half the supply pressure.

Converging-Diverging Nozzles. Consider an experiment similar to the one described above, except that a converging-diverging nozzle is to be used (Fig. 4.12).

With p_B less than p_0 by only a small amount, the flow is similar to that through a venturi passage, and it may be treated approximately as incompressible. The corresponding pressure distribution is shown by curves (i) and (ii) in Fig. 4.12.

When p_B/p_0 is reduced to the value corresponding to curve (iii), the Mach Number at the throat is unity, and no further reductions in p_T/p_0 are possible if the stream fills the passage.

We consider next the operation when the flow is entirely supersonic, corresponding to curve (iv). The value of p_B/p_0 for curve (iv) corresponds exactly to the area ratio of the nozzle, A_E/A_T, as given by the isentropic tables (in this case $A_T = A^*$, since $M_T = 1$). This is often called the "design" pressure ratio of the nozzle.

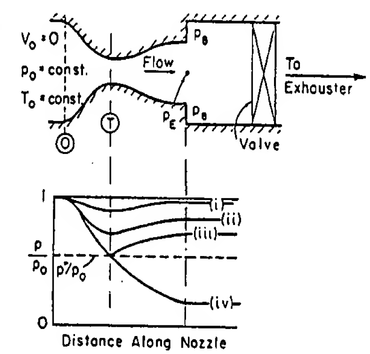


Fig. 4.12. Operation of converging-diverging nozzle at various back pressures.

No flow pattern fulfilling the conditions of isentropic and one-dimensional flow can be found which will correspond to values of p_B/p_0 between those of curves (iii) and (iv) in Fig. 4.12. One method of finding solutions for these boundary conditions is to suppose that irreversible

discontinuities involving entropy increases occur somewhere within the passage. The analysis of such discontinuities, called shock fronts, is the subject of Chapter 5. A complete discussion of the converging-diverging nozzle will, therefore, be postponed until the shock wave analysis has been presented.

4.8. Special Relations for Low Mach Numbers

In many flow problems the Mach Numbers are comparatively small, but compressibility effects cannot be entirely ignored. Using binomial expansions, the formulas of the preceding articles may be put into simple algebraic forms which are accurate and convenient for such cases.

Correction to Incompressible Pitot-Tube Formula. For example, suppose that it is desired to examine the error incurred in the computation of pressure variations when the gas is assumed incompressible. From Eqs. 4.8 and 4.14b, we find that

$$\frac{p}{p_0} = \left[1 - \frac{k-1}{2}\left(\frac{V}{c_0}\right)^2\right]^{\frac{k}{k-1}}$$

Expanding the right-hand side of this expression by the binomial theorem and rearranging, we get, if only terms up to $(V/c_0)^4$ are included,

$$\frac{p_0 - p}{\frac{1}{2}\rho_0 V^2} = 1 - \frac{1}{4}\left(\frac{V}{c_0}\right)^2 + \cdots \tag{4.25}$$

If the fluid were taken as incompressible, the right-hand side of Eq. 4.25 would reduce to unity, and the equation would be identical with Bernoulli's theorem. The departure from unity is then a measure of the error incurred in ignoring compressibility. This error in the calculation of pressure changes is shown in the following table for several values of V/c_0:

V/c_0	$\frac{p_0 - p}{\frac{1}{2}\rho_0 V^2} - 1$
0	0
0.1	−0.0025
0.2	−0.01
0.3	−0.0225
0.4	−0.04
0.5	−0.0625

Suppose that the incompressible formula were used for interpreting the reading of a pitot tube, based on the density at the stagnation pressure; at what air speed would this formula be in error by 1 per cent? In this case p is the static pressure and p_0 is the pressure measured at the mouth of the tube. If V is in error by 1 per cent, then V^2 is in error by 2 per cent. Hence we set

$$\frac{1}{4}\left(\frac{V}{c_0}\right)^2 = 0.02; \quad \text{from which} \quad \frac{V}{c_0} \cong 0.28$$

The latter figure corresponds to an air speed at normal temperatures of about 300 ft/sec. At higher speeds the error increases quite rapidly.

Isentropic Formulas in Powers of Mach Number. Expanding Eqs. 4.9, 4.14, 4.16, and 4.19 in powers of M^2 by means of the binomial theorem, the following convenient formulas for low-speed isentropic flow, valid up to orders of M^4, may be found:

$$M^* = \sqrt{\frac{k+1}{2}}\, M\left(1 - \frac{k-1}{4}M^2 + \cdots\right) \tag{4.26}$$

$$\frac{p_0 - p}{p} = \frac{kM^2}{2}\left(1 + \frac{M^2}{4} + \cdots\right) \tag{4.27}$$

$$\frac{\rho_0 - \rho}{\rho} = \frac{M^2}{2}\left(1 - \frac{kM^2}{4} + \cdots\right) \tag{4.28}$$

$$\frac{w}{A} = \sqrt{\frac{k}{R}}\frac{p_0}{\sqrt{T_0}}\, M\left(1 + \frac{k-1}{4}M^2 + \cdots\right) \tag{4.29}$$

$$\frac{A}{A^*} = \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}}\left[\frac{1}{M} + \frac{k+1}{4}M + \frac{(3-k)(k+1)}{32}M^3 + \cdots\right] \tag{4.30}$$

4.9. Deviations from Perfect Gas Laws

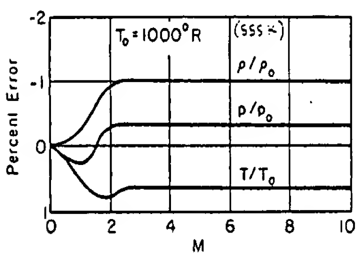
Thus far all the isentropic formulas have been based on the two assumptions of a perfect gas, namely, (i) $p = \rho RT$, and (ii) c_v = constant. In practice, either of these assumptions may be weak to some extent. For example, if the process occurs at very high temperatures but at moderate pressures, as in the case of ram jets, there may be appreciable variations in specific heat. On the other hand, if the process occurs at moderate temperatures, but is carried out at an extremely high pressure level, as in hypersonic wind tunnels, there may be significant deviations from the law $p = \rho RT$.

These two effects have been studied [1] for the isentropic flow of air. Since the analysis is a lengthy one, only the main results of practical significance are summarized here.

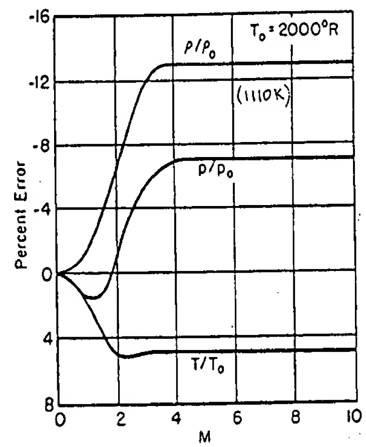
Effect of Variable Specific Heat when $p = \rho RT$. The equation of state of a perfect gas is retained at first, but the specific heat is expressed from quantum mechanics as

$$\frac{c_v}{R} = \frac{5}{2} + \left(\frac{\theta}{T}\right)^2 \frac{e^{\theta/T}}{(e^{\theta/T} - 1)^2}$$

where $\theta = 5526°R$ for air. The results are embodied in Fig. 4.13, in which are shown the per cent errors in the pressure, temperature, and density ratios incurred through use of constant rather than variable specific heats. At stagnation temperatures of 1000°R or less, the error is seen to be small for engineering purposes; but, at temperatures greater than 2000°R, the error can be substantial, especially at supersonic Mach Numbers.



(a)



(b)

Fig. 4.13. Error incurred through assumption of constant specific heats for air, with $p = \rho RT$ (after Donaldson).

(a) $T_0 = 1000°R$. (b) $T_0 = 2000°R$.

Effect of Deviations from Perfect-Gas Equation of State. Based on the use of constant specific heats but the van der Waals equation of state for air, the isentropic pressure ratio and area ratio are plotted versus M in Fig. 4.14 for several stagnation pressures. It is seen that the deviations from results obtained with the perfect-gas equation of state are negligible up to about 50 atmospheres, but are appreciable at 200 atmospheres and above.

Combined Effect of Variations in Specific Heat and Deviations from Perfect-Gas Equation of State. The simultaneous effects on pressure ratio and area ratio of both high temperature level (i.e., variations in specific heat) and high pressure level (i.e., deviations from $p = \rho RT$) are illustrated in Fig. 4.15. There is an interesting anomaly here in that the effects of pressure level are in one direction at low pressures and in the other direction at high pressures.

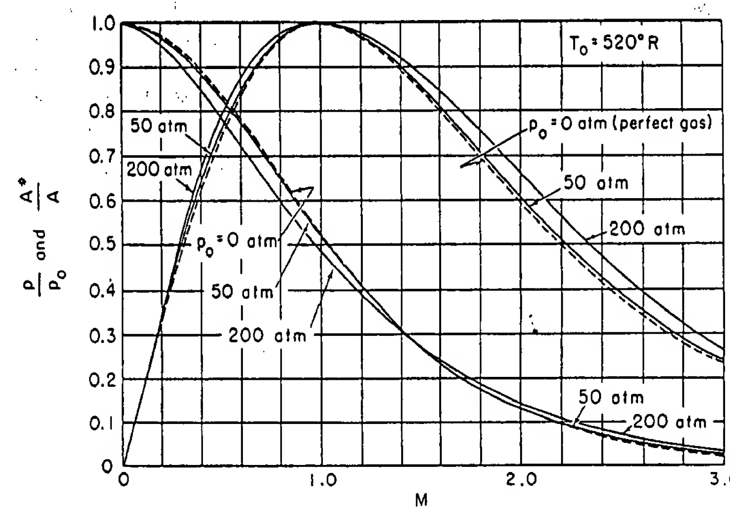


Fig. 4.14. Effect of pressure level on isentropic flow functions, using van der Waals equation for air (after Donaldson).

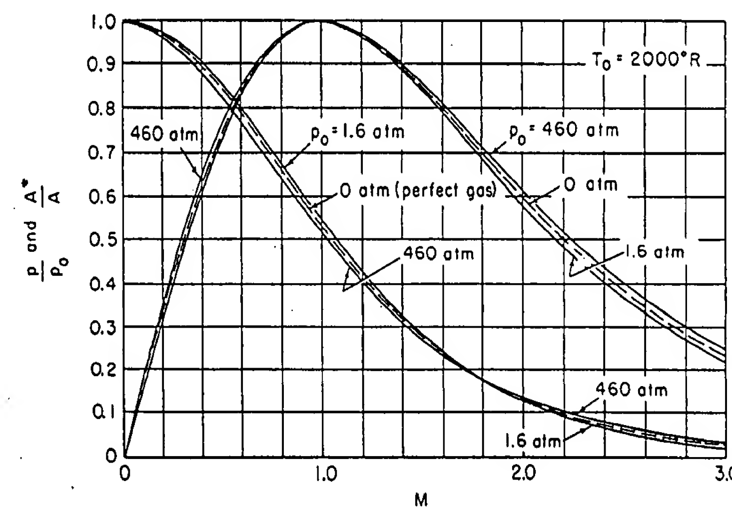


Fig. 4.15. Simultaneous effects of high pressure and high temperature on pressure ratio and area ratio (after Donaldson).

4.10. Performance of Real Nozzles

Because of frictional effects, the performance of real nozzles differs slightly from that computed with the isentropic flow relations. Since the departures from isentropic flow are usually small, the usual design procedure is based on the use of the isentropic flow formulas modified by two types of empirically determined coefficients—the nozzle efficiency and the coefficient of discharge.

Nozzle Efficiency. The term nozzle efficiency is employed primarily in turbine design where it is important to estimate accurately the average velocity leaving a nozzle passage. Consider a nozzle (Fig. 4.16) supplied with gas at low velocity and at pressure p_i and temperature T_i. The gas expands adiabatically, but with increasing entropy, to state e. If it had expanded without friction to the same final pressure, the end state would have been s. We now define the nozzle efficiency as the ratio of the exit kinetic energy to the exit kinetic energy which would be obtained in a frictionless nozzle expanding the gas to the same final pressure. With the help of the steady-flow energy equation the efficiency may be written

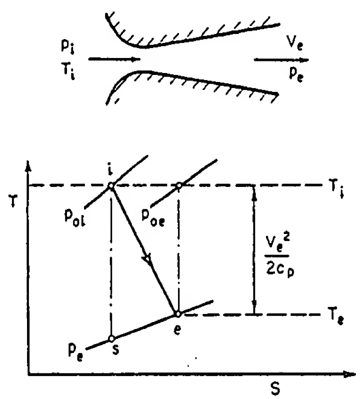


Fig. 4.16. Illustrating definition of nozzle efficiency.

$$\eta \equiv \frac{V_e^2/2}{c_p(T_i - T_s)} \tag{4.31}$$

This may be rearranged further to give

$$\eta = \frac{\dfrac{V_e^2}{2c_pT_i}\dfrac{kR}{kR}}{1 - \dfrac{T_s}{T_i}} = \frac{\dfrac{k-1}{2}\dfrac{V_e^2}{c_i^2}}{1 - \left(\dfrac{p_e}{p_i}\right)^{\frac{k-1}{k}}} \tag{4.32}$$

which is convenient for reckoning the exit velocity when the efficiency, pressure ratio, and stagnation temperature are all known.

Occasionally the term "velocity coefficient" is used, denoting the square root of the nozzle efficiency.

Frictional effects in nozzles are usually confined to thin boundary layers on the walls. Since the boundary layer thickness depends primarily on the Reynolds Number (based on some equivalent nozzle length) and on the pressure–distance curve in the nozzle, no simple expression for nozzle efficiency can be given which is applicable to all nozzles. In general, the nozzle efficiency becomes nearly unity for extremely large nozzles because the boundary layer thickness is so small compared with the size of the passage. With very small nozzles, however, the boundary layer may nearly fill the passage, and then the nozzle efficiency may drop drastically.

When well-designed nozzles with straight axes are operated at their design pressure ratio and at high Reynolds Numbers, they are found to have efficiencies ranging from 94 to 99 per cent, and even higher for sizable wind tunnel nozzles.

Well-designed turbine nozzles with curved axes have efficiencies of the order of 90 to 95 per cent when operated with suitable pressure ratios at high Reynolds Numbers.

Nozzle Discharge Coefficient. The nozzle discharge coefficient, C_w, is defined as the ratio of the actual nozzle flow to the flow calculated from the isentropic laws for the initial and final pressures of the actual nozzle.

$$C_w \equiv \frac{w}{\text{Isentropic Flow Rate}}$$

If the over-all pressure ratio of the nozzle is such that the velocity at the minimum section is subsonic, then the "isentropic flow" is reckoned in terms of the exit conditions of the nozzle. However, if the pressure ratio is such that sonic velocity prevails at the minimum section, then the "isentropic flow" is reckoned by using the formula for choking flow at the throat. These specifications apply to both converging and converging-diverging passages.

The remarks made previously concerning the factors influencing nozzle efficiency pertain also to the discharge coefficient.

For well-designed nozzles with straight axes having "pipe" Reynolds Numbers measured at the minimum area of 10^6 or more, the discharge coefficient is of the order of 0.99, but it may be considerably less for low Reynolds Numbers.

Neither the discharge coefficient nor the velocity coefficient of rounded-entrance nozzles suitably designed for the operating pressure ratio are significantly dependent upon the leaving Mach Number.

Sharp-Edged Orifice Meter. The deviation from unity of the discharge coefficient for a sharp-edged orifice meter is due primarily to the contraction (*vena contracta*) in the stream following the orifice. The contraction in turn is due to three-dimensional effects. The coefficient of contraction increases substantially as the result of compressibility effects (Fig. 4.17). It should be noted that the isentropic flow on which the discharge coefficient is based is reckoned as though a rounded-entrance converging nozzle, having the same exit area as the orifice, were supplied with gas at stagnation pressure p_1 and discharged to a region having the pressure p_2.

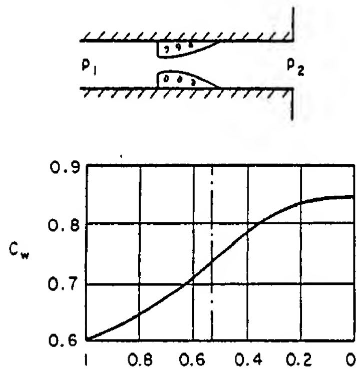


FIG. 4.17. Discharge coefficient of sharp-edged orifice meters with zero velocity of approach (after J. A. Perry).

4.11. Some Applications of Isentropic Flow

Thrust of Rocket Motor. Consider a rocket motor (Fig. 4.18a) which generates gas steadily at p_0 and T_0. The nozzle has a throat area A_t, an exit area A_e, and discharges to an atmosphere at pressure p_a. Experimental data verify that the isentropic flow equations predict, within a few per cent, the thrust produced by such a rocket.

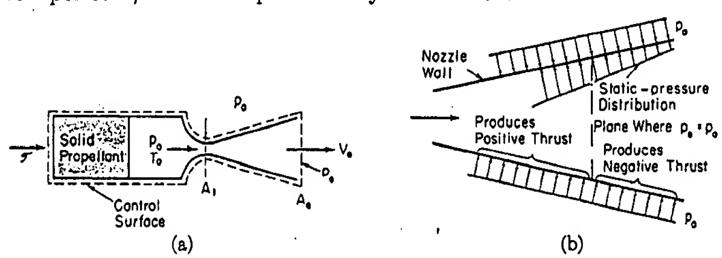


FIG. 4.18. Isentropic flow in rocket motor.

(a) Diagrammatic sketch.
(b) Pressure distributions on internal and external surfaces of diverging portion of nozzle.

Since rocket motors generate gas at about 500 psia and operate in atmospheres at 14.7 psia or less, a converging-diverging nozzle is usually used. Except under operating conditions far from the design point, sonic conditions occur at the throat, and the flow to the nozzle exit is shock-free. We shall assume these conditions for the present analysis.

In the following chapter, however, it is pointed out that a nozzle having a given area ratio and operating at certain ratios of back pressure to supply pressure cannot be isentropic because shocks are present. Such cases are not covered by the equations about to be derived.

Under the assumed conditions, the pressure ratio p_e/p_0 is fixed by the area ratio, and so the exit-plane pressure in general differs from the surrounding atmospheric pressure. Applying the momentum equation to the control volume of Fig. 4.18a, we find the thrust $\mathfrak{F}$ to be given by

$$\mathfrak{F} = wV_e + A_e(p_e - p_a)$$

which is then put into dimensionless form through division by p_0A_t:

$$\frac{\mathfrak{F}}{p_0A_t} = \frac{w}{p_0A_t}V_e + \frac{A_e}{A_t}\left(\frac{p_e}{p_0} - \frac{p_a}{p_0}\right)$$

From Eq. 4.17, for choking flow,

$$\frac{w}{A_tp_0} = \sqrt{\frac{k}{R}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}\frac{1}{\sqrt{T_0}}$$

and, from the energy equation and isentropic law,

$$V_e = \sqrt{2c_p(T_0 - T_e)} = \sqrt{2c_pT_0}\sqrt{1 - \frac{T_e}{T_0}}$$

$$= \sqrt{2c_pT_0}\sqrt{1 - \left(\frac{p_e}{p_0}\right)^{\frac{k-1}{k}}}$$

Substituting these into the thrust equation, and rearranging, there results

$$\frac{\mathfrak{F}}{p_0A_t} = k\sqrt{\frac{2}{k-1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}\sqrt{1 - \left(\frac{p_e}{p_0}\right)^{\frac{k-1}{k}}} + \frac{A_e}{A_t}\left(\frac{p_e}{p_0} - \frac{p_a}{p_0}\right) \tag{4.33}$$

Since the pressure ratio p_e/p_0 depends only on the area ratio, Eq. 4.33 indicates that the thrust for a nozzle of given size and geometry depends only on p_0 and the ratio p_a/p_0, and is independent of the temperature T_0.

EFFECT OF AREA RATIO. We now ask, for given values of A_t, p_0, and p_a, what exit area should be used in order to obtain maximum thrust? By applying the calculus to Eq. 4.33 it may be shown after a laborious calculation that $\mathfrak{F}$ is a maximum when the area ratio is chosen in such a way as to make the pressure in the exit plane exactly equal to p_a. How-

ever, this result may more easily be obtained with the simplest of physical reasoning. The net thrust on the rocket is the resultant of static pressures acting on all the surfaces of the motor. Suppose, as in Fig. 4.18b, that there is a certain exit area for which $p_e = p_a$. If the nozzle is continued beyond this point, the pressure in the nozzle will drop further, and the added piece of divergent nozzle will have negative thrust

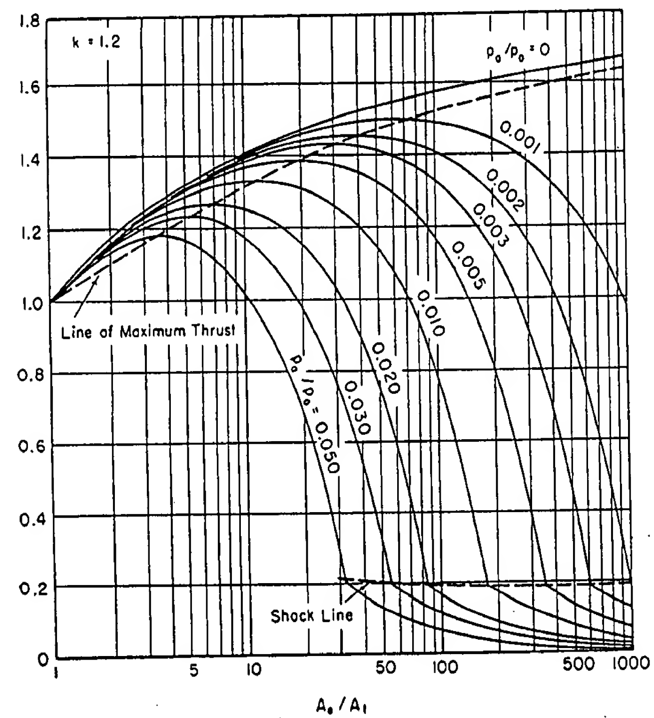


FIG. 4.19. Performance characteristics of rocket nozzle, $k = 1.2$ (after Malina).

because the internal pressure on this added piece is less than the external pressure. By similar reasoning, it follows that cutting off a piece of nozzle upstream of the plane where $p_e = p_a$ would also act to reduce the thrust. Hence we conclude that the thrust is a maximum when $p_e = p_a$. Applying this criterion to Eq. 4.33, we get

$$\frac{\mathfrak{I}_{max}}{p_0 A_t} = k\sqrt{\frac{2}{k-1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}\sqrt{1-\left(\frac{p_a}{p_0}\right)^{\frac{k-1}{k}}} \tag{4.34}$$

If the nozzle were a simple converging nozzle, A_e would equal A_t, and p_e/p_0 would be the critical pressure ratio. Making these substitutions

in Eq. 4.33 and simplifying, there is obtained

$$\frac{\mathfrak{I}_{conv}}{p_0 A_t} = 2\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} - \frac{p_a}{p_0} \tag{4.35}$$

To illustrate the effect of area ratio on nozzle thrust, we form the ratio

$$\frac{\mathfrak{I}}{\mathfrak{I}_{conv}} = \frac{\mathfrak{I}/p_0 A_t}{\mathfrak{I}_{conv}/p_0 A_t} = \frac{k\sqrt{\frac{2}{k-1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\left(1-\left(\frac{p_e}{p_0}\right)^{\frac{k-1}{k}}\right)} + \frac{A_e}{A_t}\left(\frac{p_e}{p_0}-\frac{p_a}{p_0}\right)}{2\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} - \frac{p_a}{p_0}} \tag{4.36}$$

This ratio is plotted against area ratio in Fig. 4.19 for various values of p_a/p_0, with $k = 1.2$ (typical value for rocket gas). It is seen that the curves are quite flat near their maxima, so that the area ratio need not be exactly adjusted in order to obtain substantially maximum thrust. In practice, rocket nozzles are usually designed with p_e greater than p_a, since this reduces the size of the nozzle without materially reducing the thrust.

Reynolds Number for Supersonic Wind Tunnel. Fig. 4.20 shows a supersonic wind tunnel with a test section having a Mach Number M_1

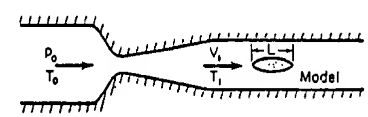


FIG. 4.20. Supersonic wind tunnel.

and in which is inserted a model of length L. We shall derive a convenient relation for the Reynolds Number of the model, based on fluid properties in the test section and on the length L. With the aid of the perfect gas laws and the isentropic flow relations, we form the expression

$$\frac{\text{Rey}}{p_0 L} = \frac{\rho_1 V_1 L/\mu_1}{p_0 L} = \frac{\rho_1 V_1}{\mu_1 p_1}\frac{p_1}{p_0} = \frac{V_1}{\mu_1 R T_1}\sqrt{\frac{kT_0}{kT_0}}\frac{p_1}{p_0} \tag{4.37}$$

$$= \frac{M_1\sqrt{\frac{k}{R}}}{\mu_1\sqrt{T_0}}\frac{\sqrt{1+\frac{k-1}{2}M_1^2}}{\left(1+\frac{k-1}{2}M_1^2\right)^{\frac{k}{k-1}}} = \frac{\sqrt{k/R}}{\mu_1\sqrt{T_0}}\frac{M_1}{\left(1+\frac{k-1}{2}M_1^2\right)^{\frac{k+1}{2(k-1)}}}$$

Now T_1, and accordingly the viscosity μ_1, are determined by the values of T_0 and M_1. Thus the Reynolds Number per unit length and per unit stagnation pressure depends, for a given gas, only on the test-section Mach Number and on the stagnation temperature. A convenient chart representing this relation for air flow is shown in Fig. 4.21.

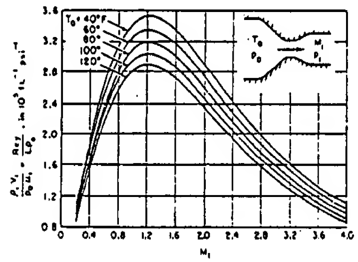


FIG. 4.21. Reynolds Number for supersonic wind tunnel (*NACA Tech. Note*, No. 1428).

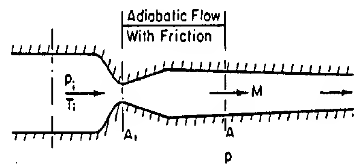


FIG. 4.22. Flow in duct with friction.

Supersonic Flow in Duct with Friction. When a duct is supplied with a supersonic flow by a converging-diverging nozzle, and when the flow in the duct is adiabatic but not frictionless, a useful expression may be obtained for determining the local Mach Number in terms of the local area and static pressure. The nomenclature is shown in Fig. 4.22.

Since the nozzle throat is choked, the mass flow through the system is given by Eq. 4.17 when modified by the discharge coefficient C_w:

$$\frac{w}{A_t} = C_w \sqrt{\frac{k}{R}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \frac{p_i}{\sqrt{T_i}}$$

Furthermore, Eq. 4.11 yields

$$\frac{w}{A} = \sqrt{\frac{k}{R}} \frac{p}{\sqrt{T_i}} \mathsf{M} \sqrt{1 + \frac{k-1}{2}\mathsf{M}^2}$$

Dividing one of these by the other, it may be shown by direct comparison that

$$C_w \frac{A_t/A}{p/p_i} = \frac{(A^*/A)_{\text{isen}}}{(p/p_0)_{\text{isen}}} \tag{4.38}$$

where $(A^*/A)_{\text{isen}}$ is the function of k and M given by Eq. 4.19, and $(p/p_0)_{\text{isen}}$ is the function of k and M given by Eq. 4.14b.

Since the type of problem discussed here arises frequently in experimental work, the quantity on the right-hand side of Eq. 4.38 is listed in the isentropic flow tables. Given the areas A_t and A and the measured

pressures p_i and p, together with the discharge coefficient C_w, the local Mach Number M may be found from these tables by a quick computation. An illustrative example is given in Chapter 6.

Eq. 4.38 typifies a technique which we shall find useful from time to time—namely, to use the tabulated functions of k and M in Appendix B not only for the particular types of flow underlying the construction of the tables, but for such other problems where identical functions of k and M appear.

REFERENCES AND SELECTED BIBLIOGRAPHY

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PROBLEMS

4.1. Consider the reversible adiabatic flow of steam through a passage of variable cross section. At the section where the velocity is zero the pressure and temperature are $p_0 = 50$ psia and $T_0 = 800°F$. Denoting the pressure at any other point in the stream by p, plot against p/p_0 the values of specific volume (ft^3/lb), velocity (ft/sec), and mass velocity (lb/ft^2 sec), for the following conditions:

(a) The properties of steam are taken from the Steam Tables of Keenan and Keyes.
(b) The steam is considered as a perfect gas, with a value of 1.3 for k.
(c) The steam is considered as incompressible with a density equal to the density corresponding to p_0 and T_0.

In the above calculations choose for the lowest value of p/p_0 the value of p which corresponds to the first appearance of moisture in part (a).

4.2. Consider the reversible, adiabatic flow of a perfect gas. Plot the values of p^*/p_0, T^*/T_0, ρ^*/ρ_0, c^*/c_0, $V_{\max}/c^*$, and $V_{\max}/c_0$, all versus k, for values of the latter between 1 and 2.

4.3. An airplane flies at an altitude of 40,000 ft (temperature = −67.0°F, pressure = 2.72 psia) with a speed of 400 mph. Neglecting frictional effects,

(a) Calculate the critical velocity of the air relative to the aircraft.
(b) Calculate the maximum possible velocity of the air relative to the aircraft.

4.4. Sketch a curve of pressure (p/p^*) versus velocity (V/V^*) for isentropic flow, paying special attention to zero or infinite slopes, direction of curvature, and points of inflection. Indicate the values of p/p^* and V/V^* at their maximum and minimum points, at points of zero or infinite slope, and at points of inflection. What is the physical significance of the tangent to the curve of p/p^* versus V/V^*?

4.5. A stream of air flows in a duct of 4 inches diameter at a rate of 2.20 lb/sec. The stagnation temperature is 100°F. At one section of the duct the static pressure is 6 psia.

Calculate the Mach Number, velocity, and stagnation pressure at this section.

4.6. A perfect gas ($k = 1.4$, $R = 100$ ft lbf/lbm°R) is supplied to a converging nozzle at low velocity and at 100 psia and 540°F. The nozzle discharges to atmospheric pressure, 14.7 psia. Assuming frictionless adiabatic flow, and a mass rate of flow of 1 lbm/sec, calculate

(a) The pressure in the exit plane, in psia
(b) The velocity in the exit plane, in ft/sec
(c) The cross-sectional area of the exit plane, in square feet

4.7. Show that for isentropic flow of a perfect gas, the pressure, temperature, and density, when made dimensionless with respect to the corresponding critical values, are given by

$$\frac{p}{p^*} = \left[\frac{k+1}{2\left(1 + \frac{k-1}{2}M^2\right)}\right]^{\frac{k}{k-1}}$$

$$\frac{T}{T^*} = \frac{k+1}{2\left(1 + \frac{k-1}{2}M^2\right)}$$

$$\frac{\rho}{\rho^*} = \left[\frac{k+1}{2\left(1 + \frac{k-1}{2}M^2\right)}\right]^{\frac{1}{k-1}}$$

4.8. Derive simplified, approximate versions of the isentropic flow relations for a perfect gas valid at Mach Numbers large compared with unity.

4.9. A pitot-static tube records a static pressure of 5.20 psig and a difference between impact pressure and static pressure of 19.42 inches of mercury. The barometer reads 29.73 inches Hg, and the stagnation temperature of the air stream is 80°F. Compute the air velocity, (a) assuming the air incompressible, and (b) assuming the air compressible.

4.10. A stream of air flowing in a duct is at a pressure of 20 psia, has a Mach Number of 0.6, and flows at a rate of 0.5 lb/sec. The cross-sectional area of the duct is one square inch.

(a) Compute the stagnation temperature of the stream in degrees F.
(b) What is the maximum percentage reduction in area which could be introduced without reducing the flow rate of the stream?
(c) For the maximum area reduction of part (b), find the velocity and pressure at the minimum area, assuming no friction and no heat transfer.

4.11. A converging nozzle with an exit area of one square inch is supplied with air at low velocity and at a pressure and temperature of 100 psia and 200°F, respectively.

Plot the mass rate of flow through the nozzle versus back pressure, assuming the flow to be isentropic.

4.12. A rocket combustion chamber is supplied with 24 lb/sec of hydrogen and 76 lb/sec of oxygen. Before entering the nozzle all the oxygen is consumed, the pressure is 23 atmospheres, and the temperature is 4960°F. Neglecting dissociation and friction, find the throat area of the nozzle required. Assume $k = 1.25$.

4.13. At a certain point in a stream tube, air flows with a velocity of 500 ft/sec and has a pressure and temperature of 10 psia and 40°F, respectively.

(a) Calculate the following quantities at a point further downstream in the stream tube where the cross-sectional area is 15 per cent smaller than at the upstream section: the stagnation pressure and temperature, the stream pressure and temperature, the velocity, the Mach Number, and the value of M^*.

(b) Compute the maximum possible reduction in area of the stream tube. For the section with the minimum area, compute the quantities listed in part (a).

4.14. When a body is placed in a stream which at infinite distance upstream is in uniform flow with free-stream conditions V_∞, p_∞, M_∞, etc., the local pressures in the neighborhood of the body are usually reported in terms of the dimensionless pressure coefficient, C_p:

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$$

(a) Show that the value of the pressure coefficient corresponding to the appearance of the critical velocity is given by

$$C_p{}^* = \frac{\left[\dfrac{2 + (k-1)M_\infty^2}{k+1}\right]^{\frac{k}{k-1}} - 1}{\dfrac{k}{2}M_\infty^2}$$

(b) Plot $\log(-C_p{}^*)$ versus $\log M_\infty$ for $k = 1.4$ and for values of M_∞ between 0.1 and 1.0.

(c) Suppose that an airplane is flying at sea level with a velocity of 500 mph. What is the maximum pressure coefficient which may be attained on the wings without the speed becoming anywhere supersonic?

4.15. Pressure coefficients, lift coefficients, drag coefficients, etc., of airfoils which are in a free stream with conditions p_∞, M_∞, etc., are usually expressed in terms of the dynamic head of the free stream. Thus

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}; \quad C_L \equiv \frac{L/A}{\frac{1}{2}\rho_\infty V_\infty^2}; \quad \text{etc.}$$

Alternate definitions for compressible flow, not usually employed, are as follows:

$$C_p' \equiv \frac{p - p_\infty}{p_0 - p_\infty}; \quad C_L' \equiv \frac{L/A}{p_0 - p_\infty}; \quad \text{etc.}$$

where p_0 is the isentropic stagnation pressure corresponding to p_∞ and M_∞.

Derive an expression for C_p'/C_p in terms of M_∞ and k. Plot C_p'/C_p versus M_∞ for $k = 1.4$ and for values of M_∞ between 0 and ∞.

4.16. (a) Show, for a source-type flow (either two- or three-dimensional) of a compressible fluid, that either supersonic or subsonic flow may subsist, but that both types may not exist together.

(b) Show that for a finite flow rate from the source there is a minimum radius within which a source-type flow pattern is impossible. What is the Mach Number at this minimum radius? Find expressions for $r/r_{\min}$ as functions of M and k for the line source (two-dimensional) and the point source (three-dimensional), where r is the radius at M and $r_{\min}$ is the minimum radius.

(c) For the four possible types of flow outside the minimum radius, depending on whether the flow is outwards (source) or inwards (sink) and on whether it is subsonic or supersonic, specify the directions of the pressure gradient and of the fluid acceleration.

4.17. Consider the vortex motion of a perfect gas in which all streamlines have the same entropy and the same stagnation-enthalpy. The equation of this motion is $Vr = \Gamma/2\pi$, where V is the tangential velocity, r is the radius of the streamline, and Γ is a constant called the circulation.

(a) Show that there is a minimum radius inside of which the vortex motion may not exist, and that this radius is given by

$$r_{\min} = \frac{\Gamma}{2\pi c^*}\sqrt{\frac{k-1}{k+1}}$$

(b) Show that the field of flow outside the minimum radius includes all Mach Numbers from zero to infinity, and that the radius corresponding to the critical velocity is given by

$$r^* = r_{\min}\sqrt{\frac{k+1}{k-1}}$$

4.18. From schlieren photographs of the flow of air through a converging-diverging nozzle it is found that the average Mach angle over the exit cross section is 40°. The measured static pressure at the exit cross section is 0.198 atm, while the pressure upstream of the nozzle, where the velocity is small, is 1.000 atm.

Calculate the ratio of the average exit kinetic energy per unit mass of the stream to the exit kinetic energy corresponding to isentropic expansion to the measured exit pressure,

(a) Using the assumption that air is a perfect gas, with $k = 1.4$
(b) Using the Air Tables of Keenan and Kaye and a measured value for the stagnation temperature, T_0, of 2400°F abs

4.19. During a reaction-stand test of a turbojet engine, measurements indicate a thrust of 1845 lb when the flow rate is 30 lb/sec. The temperature at the entrance to the thrust nozzle, where the velocity is 300 ft/sec, is 1400°F. The nozzle has no diverging section, so that the stream reaches atmospheric pressure, 14.7 psia, somewhere outside the nozzle.

Assuming no heat loss from the gas, that the direction of the air stream entering the engine is at right angles to the direction of thrust, and that the nozzle is frictionless, estimate the pressure in the exit plane of the nozzle.

4.20. Derive the following expressions for isentropic flow with the pressure ratio p/p_0 as a parameter:

$$V = \sqrt{\frac{2kR}{k-1}}\sqrt{T_0}\sqrt{1-(p/p_0)^{\frac{k-1}{k}}}$$

$$M^2 = \frac{2}{k-1}\left[(p_0/p)^{\frac{k-1}{k}} - 1\right]$$

$$M^{*2} = \frac{k+1}{k-1}\left[1-(p/p_0)^{\frac{k-1}{k}}\right]$$

$$\frac{w}{A}\frac{\sqrt{T_0}}{p_0} = (p/p_0)^{\frac{1}{k}}\sqrt{1-(p/p_0)^{\frac{k-1}{k}}}\ \sqrt{\frac{2k}{R(k-1)}}$$

$$\frac{A^*}{A} = \sqrt{\frac{2}{k-1}\left(\frac{k+1}{2}\right)^{\frac{k+1}{k-1}}}\,(p/p_0)^{\frac{1}{k}}\sqrt{1-(p/p_0)^{\frac{k-1}{k}}}$$

4.21. Derive relations between M and V/c_0 and between M and $V/V_{\max}$, for adiabatic flow of a perfect gas.

4.22. Derive a relation between M^* and the mass flow parameter

$$\frac{w}{A}\frac{\sqrt{T_0}}{p}\frac{1}{\sqrt{W}}$$

applicable to adiabatic flow of a perfect gas.

4.23. By expanding Eq. 4.11 in a power series of M with the aid of the binomial theorem, show that for low Mach Numbers the mass flow parameter may be approximated by

$$\frac{w}{A}\frac{\sqrt{T_0}}{p}\frac{1}{\sqrt{W}} = \sqrt{\frac{k}{\mathcal{R}}}\left(M + \frac{k-1}{4}M^3 + \cdots\right)$$

4.24. Derive Eqs. 4.14 without use of the steady-flow energy equation, by employing Euler's equation for frictionless flow, $dp = -\rho V\,dV$, and the perfect gas relations $p = \rho RT$, $c^2 = kRT$, and p/ρ^k = constant.

4.25. Consider a perfect gas flowing in a *constant-area* duct *adiabatically* and without friction. Changes in state come about as the result of changes in elevation in the earth's gravity field. The z-direction is away from the center of the earth, and hence gravity acts in the negative z-direction.

(a) Starting from first principles, determine by analysis the direction of change (increase or decrease) of the Mach Number, gas speed, sound speed, density, pressure, stagnation temperature, and isentropic stagnation pressure, all for *a positive increase in z*,

(i) For *subsonic* speeds
(ii) For *supersonic* speeds

(b) Is choking possible for this type of flow? Justify your answer.
(c) Considering frictionless, adiabatic gas flows for aircraft, fluid machinery, and ventilating systems, would you expect gravity effects to be significant for

(i) Speeds negligible compared with the speed of sound?
(ii) Subsonic speeds of the order of Mach Number 0.5?
(iii) Speeds very close to the local speed of sound?
(iv) Supersonic speeds of the order of Mach Number 2.0?
(v) Supersonic speeds of the order of Mach Number 10.0?

In analyzing this problem it is suggested that the governing equations be written in differential form.

4.26. Consider a nozzle with an efficiency η between the inlet and any station downstream.
(a) Derive the expression

$$\frac{p}{p_0} = \left[1 - \frac{1}{\eta}\,\frac{\frac{k-1}{2}M^2}{1 + \frac{k-1}{2}M^2}\right]^{\frac{k}{k-1}}$$

(b) Derive a corresponding expression for

$$\frac{w}{A}\frac{\sqrt{T_0}}{p_0}$$

(c) Show that Mach Number unity does not occur at the minimum area, and find the Mach Number at the throat.

4.27. Consider a supersonic nozzle constructed with a ratio of exit to throat area of 2.0. The nozzle is supplied with air at low speed at 100 psia and 140°F. The over-all nozzle efficiency from inlet to exit is 90 per cent, but the flow is isentropic to the throat.

Calculate the pressure, velocity, and Mach Number at exit, and compare with the corresponding values for isentropic flow.

4.28. A tank having a volume of 100 ft³ is initially filled with air at 100 psia and 140°F. Suddenly the air is allowed to escape to the atmosphere (14.7 psia) through a frictionless converging nozzle of one-inch diameter. It is agreed to assume that the flow is quasi-steady, i.e., that the steady flow equations may be applied to the nozzle at any instant of time. Furthermore, the tank is to be considered as insulated perfectly against heat conduction and as having no heat capacity.

Plot the pressure in the tank versus elapsed time.

4.29. A supersonic nozzle with a throat area of 1 sq in. discharges air into a duct having a cross-sectional area of 2 sq in. The supply pressure is 100 psia, and the nozzle has a discharge coefficient of 0.98. At one section of the duct the pressure is 14.2 psia. Calculate the Mach Number and isentropic stagnation pressure at this section.

4.30. Consider the isentropic flow of a highly compressible liquid having a pressure–density relation given by

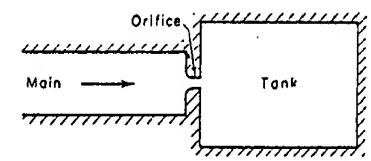
$$\beta = \rho\left(\frac{\partial p}{\partial \rho}\right),$$

where β is a constant.
(a) Show that

$$p_0 - p^* = \beta \ln 2$$

where p_0 is the stagnation pressure and p^* is the critical pressure.
(b) Derive expressions for p/p_0 and A/A^* in terms of M and β.

4.31. A large main is connected to an evacuated tank with a volume of 10 ft³ by means of a rounded-entrance, converging nozzle having a diameter of 0.01 in. Initially, a diaphragm over the orifice seals the tank from the main. The air in the main is at 100 psia and 70°F. The diaphragm is suddenly broken and air rushes into the tank. Estimate the time required for the pressure in the tank to reach 25 psia, based on the following assumptions:



PROB. 4.31.

(i) The flow is quasi-static.
(ii) There is no heat conduction from the tank to the air.
(iii) The pressure and temperature in the main are constant.

Under what circumstances will these assumptions lead to accurate results?

4.32. Show that the coefficient of contraction for a Borda re-entrant orifice is given by

$$\frac{1}{kM^2}\left[\left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}} - 1\right]$$

where M is the Mach Number of the jet.

Note that the coefficient goes from 0.50 at $M = 0$ to approximately 0.64 at $M = 1$, and that the percentage change is of the same order of magnitude as that for ordinary sharp-edged orifices.